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EXTERNAL INFLUENCES ON THE GROWTH AND DECAY OF ONE-DIMENSIONAL SHOCK WAVES IN ELASTIC NON-CONDUCTORS

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1. INTRODUCTION

IN THIS paper, we study the influence of discontinuous external body force and external radiation on the behavior of shock waves in elastic materials which do not conduct heat. To this end, we derive a differential equation relating the strain ε^- and grain gradient ε_X^- behind a shock when the material region ahead is unstrained and at constant entropy. First, this equation implies the existence of an externally induced critical strain gradient λ, proportional to the external body force and external radiation behind the wave. Next, for a compression shock $|\varepsilon^-|$ will, in general, increase, decrease, or remain constant accordingly as ε_X^- is greater than, less than, or equal to λ. These results should be compared with those given by Chen and Gurtin [1], who studied the behavior of shock waves in elastic non-conductors of heat, assuming that the external body force and external radiation are absent. In their paper, they showed that the conditions under which a shock will grow or decay are, in general, qualitatively the same as in the purely mechanical theory. Finally, we outline a particular experiment for which the external radiation is indeed discontinuous across the shock and give a procedure for determining λ as a function of ε^-. In principle, this experiment can be readily conducted in the laboratory.

2. CONSTITUTIVE ASSUMPTIONS

In this article, we consider the one-dimensional motion of a homogeneous elastic non-conductor of heat defined by the constitutive relations

$$\begin{aligned} e &= \hat{e}(\varepsilon, \eta), \\ \sigma &= \hat{\sigma}(\varepsilon, \eta), \\ \theta &= \hat{\theta}(\varepsilon, \eta), \end{aligned} \tag{2.1}$$

where e is the internal energy, σ the stress, θ the absolute temperature, ε the strain and η the entropy. The strain is defined by†

$$\varepsilon = u_X, \tag{2.2}$$

where $u = u(X, t)$ is the displacement at time t of the material point X, identified with its position in a fixed homogeneous reference configuration with density ρ_R. The response functions $\hat{e}$, $\hat{\sigma}$, $\hat{\theta}$ are not independent. They are related by the stress-relation and the temperature-relation, i.e.

$$\hat{\sigma} = \hat{e}_\varepsilon, \qquad \hat{\theta} = \hat{e}_\eta. \tag{2.3}$$

† Subscripts denote partial differentiation with respect to the corresponding variable.

We shall assume that $\hat{e}$ is of class C^3, hence, by (2.3), $\hat{\sigma}$ and $\hat{\theta}$ are of class C^2. Finally, we call the quantities,

$$E = \hat{\sigma}_\varepsilon, \qquad G = \hat{\sigma}_\eta, \tag{2.4}$$

the *tangent modulus* and *stress-entropy* modulus, respectively, and assume that

$$E(\varepsilon, \eta) > 0, \qquad G(\varepsilon, \eta) \neq 0. \tag{2.5}$$

3. GENERAL PROPERTIES OF SHOCK WAVES

We assume that the motion contains a shock moving with *intrinsic velocity*

$$U(t) = \frac{\mathrm{d}Y(t)}{\mathrm{d}t}, \tag{3.1}$$

where $Y(t)$ is the material point at which the wave is to be found at time t, i.e.

(i) If f denotes either ε, $\dot{u}$ or η, then f, $\dot{f}$ and f_X have jump discontinuities across the waves, but are continuous everywhere else.

We further assume that

(ii) The external body force b and the external radiation r have jump discontinuities across the wave, but are continuous everywhere else.

Thus, by (2.1), (i) also holds with f equal to e, σ or θ. Furthermore, we have the following compatibility conditions:

$$\begin{aligned} U[\varepsilon] &= -[\dot{u}], \\ \frac{\mathrm{d}[f]}{\mathrm{d}t} &= [\dot{f}] + U[f_X]. \end{aligned} \tag{3.2}$$

Here, we have used the notation $[f]$ to denote the jump in a function $f(X, t)$, i.e. $[f] = f^- - f^+$ with $f^\pm = \lim_{X \to Y \pm (t)} f(X, t)$. We assume that $U(t) > 0$, then f^- and f^+ are, respectively, the limiting values of f immediately behind and just in front of the wave.

Now, balance of momentum and balance of energy imply that

$$\begin{aligned} \sigma_X + \rho_R b &= \rho_R \ddot{u}, \\ \dot{e} &= \sigma\dot{\varepsilon} + \rho_R r, \end{aligned} \tag{3.3}$$

for all $X \neq Y(t)$, and

$$\begin{aligned} [\sigma] &= -\rho_R U[\dot{u}], \\ [\sigma_X] + \rho_R[b] &= \rho_R[\ddot{u}], \\ -U[e + \tfrac{1}{2}\rho_R \dot{u}^2] &= [\sigma\dot{u}], \\ [\dot{e}] &= [\sigma\dot{\varepsilon}] + \rho_R[r]. \end{aligned} \tag{3.4}$$

Formula (3.4_1) with (3.2_1) yields the well-known result

$$U^2 = \frac{[\sigma]}{\rho_R[\varepsilon]} \tag{3.5}$$

for the intrinsic velocity of the shock, while (3.4_2) and (3.2_2) with $f = \dot{u}$ and $f = \varepsilon$ imply that

$$2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}+[\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} = U^2[\varepsilon_X]-\frac{1}{\rho_R}[\sigma_X]-[b]. \tag{3.6}$$

In view of the constitutive relation (2.1_2) and (2.4), (3.6) may be expressed in the alternate form

$$2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}+[\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} = U^2[\varepsilon_X]-\frac{1}{\rho_R}\{[E\varepsilon_X]+[G\eta_X]\}-[b]. \tag{3.7}$$

Finally, (3.4_4) with (2.1_1) and (2.3) implies that

$$[\theta\dot{\eta}] = \rho_R[r]. \tag{3.8}$$

4. SHOCK WAVE ENTERING MATERIAL IN A HOMOGENEOUS STATE

Henceforth, we shall assume that the material ahead of the wave is in a state of zero strain and constant entropy η_0. Thus

$$\varepsilon^+ = \dot{u}^+ = \varepsilon_X^+ = \dot{\varepsilon}^+ = \dot{\eta}^+ = \eta_X^+ = \ddot{u}^+ = 0, \tag{4.1}$$

so that

$$[f] = f \tag{4.2}$$

for $f = \varepsilon, \dot{u}, \varepsilon_X, \dot{\varepsilon}, \dot{\eta}, \eta_X$ or $\ddot{u}$. Furthermore, the above assumption together with the balance laws (3.3) implies that $b = r = 0$ for all $X > Y(t)$.† Consequently,

$$b^+ = r^+ = 0. \tag{4.3}$$

Hence, by (4.2) and (4.3), formulae (3.7) and (3.8) may be expressed in the alternate forms

$$\begin{aligned} 2U\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}+[\varepsilon]\frac{\mathrm{d}U}{\mathrm{d}t} &= \left(U^2-\frac{E^-}{\rho_R}\right)\varepsilon_X^- -\frac{G^-}{\rho_R}\eta_X^- -b^-, \\ \theta^-\dot{\eta}^- &= \rho_R r^-, \end{aligned} \tag{4.4}$$

where

$$E^- = E(\varepsilon^-,\eta^-), \qquad G^- = G(\varepsilon^-,\eta^-).$$

Now, (3.4_3) with (3.2_1) and (3.5) reduces to

$$[e]-\tfrac{1}{2}(\sigma^- +\sigma^+)[\varepsilon] = 0, \tag{4.5}$$

which is the well-known Hugoniot relation. On the other hand, (3.4_4) and (3.2_2) with $f = \varepsilon$ imply that

$$[\dot{e}] = \sigma^-\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}-\sigma^- U\varepsilon_X^- +\rho_R r^-, \tag{4.6}$$

† Within the context of the present theory, it is difficult for one to give examples for which both $b = r = 0$ for all $X > Y(t)$ and $b \neq 0$, $r \neq 0$ for all $X < Y(t)$. However, there is indeed an example for which $b = 0$ for all X; and, while $r = 0$ for all $X > Y(t)$, $r \neq 0$ for all $X < Y(t)$. The relevant details will be given in Section 6.

and (2.1_1) with (2.3) yields the relation

$$[e_X] = \sigma^- \varepsilon_X^- + \theta^- \eta_X^-. \tag{4.7}$$

Thus, (3.2_2) with $f = e$, (4.6) and (4.7) imply that

$$\frac{\mathrm{d}[e]}{\mathrm{d}t} = \sigma^- \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + \rho_R r^- + \theta^- U\eta_X^-. \tag{4.8}$$

Differentiating (4.5) with respect to t and combining the resulting relation with (4.8), we have

$$\theta^- U\eta_X^- = -\frac{[\sigma]}{2}\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + \frac{[\varepsilon]}{2}\frac{\mathrm{d}\sigma^-}{\mathrm{d}t} - \rho_R r^-. \tag{4.9}$$

By (2.1_2), (2.4), (3.2_2), with $f = \eta$, and (4.4_2),

$$\frac{\mathrm{d}\sigma^-}{\mathrm{d}t} = E^- \frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} + G^-\left(\frac{\rho_R r^-}{\theta^-} + U\eta_X^-\right), \tag{4.10}$$

which together with (3.5) and (4.9) imply that

$$\eta_X^- = \frac{E^-(1-\mu)}{G^- U(2\tau-1)}\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} - \frac{\rho_R r^-}{U\tau G^-[\varepsilon]}, \tag{4.11}$$

with

$$\mu = \frac{\rho_R U^2}{E^-}, \qquad \tau = \frac{\theta^-}{G^-[\varepsilon]}. \tag{4.12}$$

Differentiating (3.5) with respect to t and utilizing (4.10), (4.11) and (4.12) we find that

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\tau(1-\mu)E^-}{\rho_R U(2\tau-1)[\varepsilon]}\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}. \tag{4.13}$$

Finally, by (4.4_1), (4.11) and (4.13),†

$$\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t} = -\frac{U(1-\mu)(2\tau-1)}{(3\mu+1)\tau-(3\mu-1)}(\varepsilon_X^- - \lambda), \tag{4.14}$$

with

$$\lambda = \frac{\rho_R}{E^-(1-\mu)}\left(\frac{G^- r^-}{\theta^- U} - b^-\right), \tag{4.15}$$

and, by (3.2_2) with $f = \eta$, (4.4_2) and (4.11_1),‡

$$\frac{\mathrm{d}[\eta]}{\mathrm{d}t} = \frac{E^-(1-\mu)}{G^-(2\tau-1)}\frac{\mathrm{d}[\varepsilon]}{\mathrm{d}t}. \tag{4.16}$$

† Neglecting the influence of thermodynamics, a relation of this type has been derived by Harris [2] for an ideal gas, by Duvall and Alverson [3] for a nonlinear Maxwell material and by Chen and Gurtin [4] for a general nonlinear viscoelastic material. Besides their earlier work on shock waves in elastic non-conductors, Chen and Gurtin [5] also studied the behavior of shock waves in fluids with internal state variables. None of the above papers included the influence of discontinuous external body force and external radiation. A linearized version of this relation has been obtained directly by Walsh [6]. He assumes from the outset that $[b]$, $[r]$, $[\varepsilon]$, $[\varepsilon_X]$ and $[\dot{\varepsilon}]$ are all small compared with unity. Thus, his linear theory need not be relevant insofar as the example, given Section 6, is concerned.

‡ This same result is valid when the external body force and external radiation are absent or continuous. It is not given in the paper by Chen and Gurtin [1].

We call the quantity λ the *externally induced critical strain gradient*. In practical applications, it is possible that either the external body force or the external radiation may be absent, then λ will have certain obvious reduced forms.

5. COMPRESSIVE SHOCK WAVES

We now consider a compression shock for which

$$[\varepsilon] = \varepsilon^- < 0. \tag{5.1}$$

We assume that the isentropic stress–strain law is concave from below, i.e.

$$\hat{\sigma}_{\varepsilon\varepsilon}(\varepsilon, \eta) < 0 \tag{5.2}$$

for all $\varepsilon \leq 0$ and all η, and that in the $(\varepsilon^-, \sigma^-)$-plane the Hugoniot relation (4.5) can be represented in the form $\sigma^- = \sigma_H(\varepsilon^-)$, then, by (2.5) and (5.1), we can establish the following:

(i) The intrinsic velocity of the shock is supersonic with respect to the material ahead and subsonic with respect to the material behind, i.e.

$$E(0, \eta_0) < \rho_R U^2 < E(\varepsilon^-, \eta^-). \tag{5.3}$$

(ii) Along the entire Hugoniot curve the entropy increases with decreasing strain.

We omit giving the somewhat lengthy derivations of the preceding results. In fact, they follow analogously from the known derivations of the corresponding results for compression shocks in elastic fluids.† In the present context, however, external radiation and external body force are present.

In view of (2.5_1) and (5.3) we see that μ, defined by (4.12_1), has the property

$$0 < \mu < 1. \tag{5.4}$$

Next, in view of the preceding result (ii) and our assumption that the material ahead of the wave is in a state of zero strain and constant entropy η_0, we must have‡

$$\operatorname{sgn} \frac{d[\eta]}{dt} = \operatorname{sgn} \frac{d|[\varepsilon]|}{dt}$$

This, when taken together with (4.16) implies that§

$$\frac{E^-(1-\mu)}{G^-(2\tau-1)} < 0 \tag{5.5}$$

provided that $d|[\varepsilon]|/dt \neq 0$; for this reason we assume that (5.5) holds. Thus, by (2.5_1), (5.4) and (5.5), we have

$$G^- < 0 \Leftrightarrow \tau > \tfrac{1}{2} \tag{5.6}$$

and, by (4.12_2) and (5.2),

$$G^- > 0 \Leftrightarrow \tau < 0. \tag{5.7}$$

Now, in view of (5.2), (5.4), (5.6) and (5.7), we have the following important results from the shock amplitude equation (4.14):

† Compare Bethe [7], Weyl [8], Courant and Friedrichs [9, pp. 141–148] and Serrin [10, Section 56].

‡ It is of importance to point out that this result is not true if the material ahead of the shock is *not* in a state of zero strain and constant entropy.

§ A different argument leading to the quality (5.5) has also been obtained by Nunziate, private communication (1971).

Consider a shock propagating in a material that is in a state of zero strain and constant entropy, and assume that (5.1) *holds.*

1. *If* $G^- < 0$ *or if* $G^- > 0$ *and* $\tau < \frac{3\mu-1}{3\mu+1}$, *then*

$$\begin{aligned} \varepsilon_X^- < \lambda &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} < 0, \\ \varepsilon_X^- > \lambda &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} > 0. \end{aligned} \tag{5.8}$$

2. *If* $G^- > 0$ *and* $\tau > \frac{3\mu-1}{3\mu+1}$, *then*

$$\begin{aligned} \varepsilon_X^- < \lambda &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} > 0, \\ \varepsilon_X^- > \lambda &\Leftrightarrow \frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} < 0. \end{aligned} \tag{5.9}$$

3. *If* $\varepsilon_X^- = \lambda$, *then*

$$\frac{\mathrm{d}|\varepsilon^-|}{\mathrm{d}t} = 0. \tag{5.10}$$

For most materials $G^- < 0$, therefore we would expect that (5.8) and (5.10) will hold in most situations.

Finally, in the application of the present theory, we expect that $b = 0$ for all X and the material is absorbing heat. Thus $r^- > 0$, and, by (2.5_1), (4.15) and (5.4), we conclude that

$$\lambda < 0 \tag{5.11}$$

for most materials.

6. A POSSIBLE EXPERIMENT AND ITS CONSEQUENCES

As we have remarked earlier, it is difficult, within the context of the present theory, for one to furnish examples for which both the external body force and external radiation are absent in the material region ahead of the wave, and are non-zero in the material region behind the wave. Here, we give an example for which there is no external body force throughout the material, and, while the external radiation is absent ahead of the wave, it is indeed non-zero behind the wave.

Consider a plate of transparent material. Direct a high energy laser beam on one of its surfaces. Of course, the beam passes through the material without being absorbed. Now, hit the opposite surface of the plate with another plate at sufficiently high velocity so as to generate a compression shock, and such that the material region behind the wave becomes opaque.† Thus, the material region behind the wave will absorb the laser radiation, while

† For example, the opacity of ionic crystals of sodium chloride and potassium chloride during shock compression has been studied by Kormer *et al.* [11] and Kormer *et al.* [12]. Barker and Hollenbach [13] concluded that sapphire possibly becomes opaque during shock compression at stresses between 130–150 kbar. Additional relevant references are given in the review article, Doran and Linde [14].

the material region ahead of the wave, remaining transparent, will not absorb any, i.e. there is a discontinuous external radiation accompanying the moving shock. In principle, this experiment can be conducted in the laboratory.

First, we note that the radiation absorbed by the material behind the wave depends on the degree of its opacity. Thus, r^- would vary for shocks of various strengths; it may also vary as a shock amplifies or decays. Second, when $\varepsilon_X^- = \lambda$, with

$$\lambda = \frac{G^- \rho_R r^-}{\theta^- U E^-(1-\mu)}, \tag{6.1}$$

the shock is steady, i.e. $(d|\varepsilon^-|)/dt = 0$. Thus, in view of our assumption regarding the Hugoniot relation which implies the existence of a relation expressing η^- as a function of ε^- and for instances for which r^- depends on ε^-, we can experimentally determine λ as a function of the jump in strain by measuring the strain gradient ε_X^- behind the wave for steady shocks of various strengths.† Finally, if the material properties and λ are known, then, by (6.1), we can determine r^-.

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REFERENCES

[1] P. J. Chen and M. E. Gurtin, The growth of one-dimensional shock waves in elastic nonconductors. *Int. J. Solids Struct.* **7**, 5–10 (1971).
[2] A. J. Harris, The Decay of Plane, Cylindrical and Spherical Shock Waves, in Underwater Explosion Research, Vol. 1, entitled: The Shock Wave, A Compendium of British and American Reports, Office of Naval Research, Washington, D.C. (1950).
[3] G. D. Duvall and R. C. Alverson, Fundamental Research in Support of Vela-Uniform, Semiannual Technical Summary Report No. 4, Stanford Research Institute, Menlo Park, California (1963).
[4] P. J. Chen and M. E. Gurtin, On the growth of one-dimensional shock waves in materials with memory. *Archs Ration. Mech. Analysis* **36**, 33–46 (1970).
[5] P. J. Chen and M. E. Gurtin, On the growth and decay of one-dimensional shock waves in fluids with internal state variables, to be published.
[6] E. K. Walsh, Induced one-dimensional waves in elastic non-conductors. *J. appl. Mech.* **34**, 937–941 (1967).
[7] H. Bethe, The Theory of Shock Waves for an Arbitrary Equation of State, Office of Scientific Research and Development, Division B, Report No. 545 (1942).
[8] H. Weyl, Shock waves in arbitrary fluids. *Communs pure appl. Math.* **2**, 103–122 (1949).
[9] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*. Interscience (1948).
[10] J. Serrin, Mathematical Principles of Classical Fluid Mechanics, *Handbuch der Physik*, Vol. VIII/1, edited by S. Flügge. Springer-Verlag (1959).
[11] S. B. Kormer, M. V. Sinitsyn, G. A. Kirillov and V. D. Urlin, Experimental determination of temperature in shock-compressed NaCl and KCl and of their melting curves at pressures up to 700 kbar. *Soviet Phys. JETP* **21**, 689–700 (1965).
[12] S. B. Kormer, M. V. Sinitsyn, G. A. Kirillov and L. T. Popova, Experiment determination of the light absorption coefficient in shock-compressed NaCl. The absorption and conductivity mechanism. *Soviet Phys. JETP* **22**, 97–105 (1966).
[13] L. M. Barker and R. E. Hollenbach, Shock wave studies of PMMA, fused silica and sapphire. *J. appl. Phys.* **41**, 4208–4226 (1970).
[14] D. G. Doran and R. K. Linde, Shock effects in solids. *Solid St. Phys.* **19**, 229–290 (1966).

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† Compare Chen and Gurtin [4, 5].